2 Some more things about locally convex tvs

Definition 2.1 *A subset A of a tvs is called* **s-bounded** *iff, for any open neighborhood U of 0, there is a $k \in K$ with $A \subset kU$.*

Theorem 2.2 *Compact sets are s-bounded, finite unions of s-bounded sets are s-bounded, Cauchy sequences are s-bounded.*

Proof. The proof of the first two assertions in an easy **exercise (1)**. Now, given a Cauchy sequence a_n and a zero neighborhood N, there is a starshaped zero neighborhood W with $W + W \subset N$, and there is an $N \in \mathbb{N}$ with $a_m - a_N \in W$ for all $m > N$. Now we can find an $r > 0$ with $ra_N \in W$. Then $a_n \in r^{-1}W + W \subset r^{-1}V$. The image of the first N terms is compact, hence s-bounded. $\square$

Sometimes, one has to extremize a functional on a compact convex subset. To establish a general theory of the critical points in such sets let us first introduce some abbreviation: For two points x, y of a tvs E, let $[x, y] := \{tx+(1-t)y | t \in [0, 1]\}$ and $(x, y) := \{tx + (1 - t)y | t \in [0, 1]\}$.

Definition 2.3 *Let E be a locally convex tvs and K a convex subset of E. A nonempty, convex and compact subset S of K is called* **face of** *K iff for all $x, y \in K$ we have*

$$S \cap (x, y) \neq \emptyset \Rightarrow [x, y] \subset S.$$

Faces of K consisting of one element only are called **extreme points of** *K.*

Example: Given a real locally convex tvs E, a nonempty convex compact subset K of E and an $f \in E^* = CL(E, \mathbb{R})$, then $m_K(f) := \{k \in K | f(k) = max_{x \in K} f(x)\}$ is a face of K.

Theorem 2.4 *Each face of a nonempty compact convex subset K of a locally convex real tvs E contains an extreme point of K.*

Proof. First order the faces partially by inverse inclusion: $S_1 < S_2 :\Leftrightarrow S_1 \supset S_2$. Then use Zorn's Lemma whose conditions are easily checked. Therefore there is a maximal element M. If it has more then one element, $x, y \in M$, $x \neq y$, then with the Separation theorem we can find an $f \in E^*$ with $f(x) \neq f(y)$. Then $M \neq m_M(f)$, the latter defined as in the previous example, and as $m_M(f)$ is a face, we get a contradiction. $\square$

Theorem 2.5 *Let E be a real locally convex tvs, let $f \in E^* = CL(E, \mathbb{R})$. Then for every compact convex subset K of E, the maximum of f in K is attained at an extreme point.*

Proof. Consider the example and the preceding theorem. $\square$

In locally convex tvs, to a convex zero-neighborhood U one can associate a characteristic continuous functional μ_U, the so-called **Minkowski functional of** U. It is defined by

$$\mu_U(V) := inf\{\lambda > 0 | \frac{1}{\lambda} \cdot v \in U\}.$$

These Minkowski functionals are subadditive, as for $\frac{1}{\lambda} f, \frac{1}{\mu} g \in U$ we have also $\frac{1}{\lambda+\mu}(f + g) \in U$ as a convex combination. Continuity is an easy consequence of subadditivity. Observe that $\mu_U(x) < 1$ if and only if $x \in V$.

Theorem 2.6 (Krein-Milman Theorem) *Let E be a locally convex tvs and K a nonempty convex compact subset of E, let* Ext *be the set of the extreme points of K. Then we have $K = \overline{\mathrm{conv}(\mathrm{Ext})}$.*

Proof. Define $A := \overline{\mathrm{conv}(\mathrm{Ext})}$, we have to show $K \subset A$. W.r.o.g. assume that $0 \in A$. Choose $x \in E \setminus A$, we have to show $x \in E \setminus K$. There is a convex zero neighborhood U with $x+2U \subset E \setminus A$. As $0 \in A$, $A+U$ is a convex zero neighborhood containing U, and $\mu_{A+U} \leq \mu_U$. Then $\mu_{A+U}|_{\mathbb{R}x}$ is linear and, by the Hahn-Banach Extension Theorem, has an extension $m \in L(E, \mathbb{R})$ with $m \leq \mu_U$. Therefore m is continuous. $x \notin A + 2U$ implies $m(x) = \mu_{A+2U}(x) > 1$. Now from Theorem 2.5 we get

$$\mathrm{sup}_{p \in K} m(p) = \mathrm{max}_{p \in \mathrm{Ext}} m(p) \leq \mathrm{max}_{p \in A} m(p) \leq 1 < m(x),$$

thus $x \in E \setminus K$. $\square$

Definition 2.7 *A subspace A of a vector space V is called* **complemented in** V *iff there is another subspace B of V such that every vector $v = V$ has a unique decomposition $v = a + b, a \in A, b \in B$. A subspace G of a tvs E is called* **topologically complemented in** E *if there is another subspace H of E such that the map $\Phi : G \times H \to E, \Phi((g, h)) := g + h$ is a homeomorphism, i.e. if E is homeomorphic to the topological direct sum $G \oplus H$. In this case we call H a* **topological complement of** G **in** E.

There is always a complement for any subspace A of a vector space V: Let $B_1 = \{x_a\}$ be a Hamel basis for A and $B_2 = \{y_b\}$ be a Hamel basis for V/A. Then choose an element z_b for every equivalence class y_b, and then it is straightforward to express an arbitrary vector as a linear combination of the x_a and z_b and to see that the union forms a Hamel basis, Thus one can define the complement as the linear hull of the z_b.
But the complement cannot always be chosen topological; we will see an example in the part about Fréchet spaces. As there is a homeomorphism from G to $G \oplus \{0\} \subset G \oplus H$, a topological complemented subspace is always closed. Likewise, it is easy to see that G is topologically complemented in E if and only if there is a continuous projection $\pi : E \to G$.

Often (e.g. in the overnext theorem and in general when dealing with fix point theorems) graphs of continuous maps enter as a natural object to study. It is useful to know that almost always they are closed:

Theorem 2.8 *Let X be a topological space, Y a Hausdorff space, $f : X \to Y$ continuous, then $G_f := \{(x, f(x))|\}$, as a subset of $X \times Y$ with the product topology, is closed.*

Proof. Put $U := X \times Y \setminus G_f$ and pick $(x, y) \in U$. Then $y \neq f(x)$, and we can find disjoint neighborhoods $V \ni y$, $W \ni f(x)$. By continuity of F we find a neighborhood N of X with $f(N) \subset W$, then $N \times V$ is a neighborhood of (x, y) in $X \times Y$ disjoint from G. $\square$

Theorem 2.9 (Schauder's fix point theorem) *Let V be a locally convex tvs, $\emptyset \neq C \subset V$ compact and convex, and $f : C \to C$ continuous, then C contains a fix point p of f, i.e. $p \in C$ with $f(p) = p$.*

Proof. Assume the contrary: that f is free of fix points. Then its graph $G := \{(x, f(x))|x \in C\} \subset V \times V$, equipped with the subset topology of the product topology in $V \times V$, is compact and disjoint from the diagonal Δ in $V \times V$, thus there is a convex starshaped zero neighborhood W in V such that $G + W \times W$ is still disjoint from Δ **(why? Exercise(2))**. Therefore, for all $x \in C$ we have

$$f(x) \notin x + W. \tag{1}$$

Now consider the Minkowski functional μ_W and define $a \in C^0(V, \mathbb{R})$ by $a(x) := \max\{0, 1 - \mu_W(x)\}$. Now choose $x_1, ...x_n \in C$ such that $\{x_1 + W, ...x_n + W\}$ is an open covering of C, define $a_i(x) := a(x - x_i)$ and

$$b_i = a_i / \sum_{j=1}^{n} a_j.$$

This is well-defined as the denominator is always positive. Now consider $H := \mathrm{conv}\{x_1, ...x_n\}$ which is a subset of V homeomorphic to a compact finite-dimensional simplex. Define $g \in C^0(K, H)$ by $g(x) := \sum_{i=1}^{n} b_i(x) \cdot x_i$. Then Brouwer's fix point theorem applied to $g \circ f : K \to H$ implies that there is an $p \in H$ with $g(f(p)) = p$. As $\mathrm{supp}(b_i) \subset x_i + W$, for every $x \in K$ we have

$$x - g(x) = \sum_{i=1}^{n} b_i(x)(x - x_i),$$

thus $x - g(x)$ is a convex combination of the $x - x_i \in W$, so for every $x \in C$ we have $x - g(x) \in W$. Applying that to $x = f(p)$ we get

$$f(p) \in g(f(p)) + W = p + W$$

in contradiction to the condition 1. $\square$

Theorem 2.10 (Schauder reloaded) *Let V be a locally convex tvs, $\emptyset \neq K \subset V$* closed *and convex, and $f : K \to K$ continuous* and with precompact image*, then K contains a fix point p of f, i.e. $p \in K$ with $f(p) = p$.*

Proof: Exercise (3). $\square$

3 Metric vector spaces and Fréchet spaces

First we recall that a Fréchet space is just a locally convex complete metrizable tvs. We want to give some equivalent condition to metrizability. To this aim, let us call a set B of open subsets of a topological space T a **local base at** $p \in T$ iff every neighborhoof of p contains an element of B. Then we have the following theorem:

Theorem 3.1 *If V is a Hausdorff tvs with a countable local base at 0, then there is a compatible translation-invariant metric on V whose balls are starshaped. If V is locally-convex, such a metric can be chosen such that it satifies the additional condition that its balls are convex. Conversely, if V is metrizable, it has a countable local base at 0.*

Proof. The last part is trivial due to the existence of the balls with radius $1/n$. For the first statement, construct inductively a new local base B with $B_{n+1} + B_{n+1} + B_{n+1} + B_{n+1} \subset B_n$. We will use dyadic numbers, the analogue to floating-point decimal numbers, but for the base 2. Formally, the set D of dyadic numbers is defined as the subset of $\mathbb{Q}$ consisting of numbers $r = \sum_{i=1}^{\infty} s_n(r) \cdot 2^{-n}$ for a sequence $\{s_n(r)\}_{n \in \mathbb{N}}$ with values in $\{0, 1\}$ and with compact support, i.e. only finitely many of its terms are nonzero. Obviously, all dyadic numbers are contained in $[0, 1]$.

Define $\overline{D} := D \cup \{1\}$. Now by means of $\overline{D}$ and the elements of the base we define subsets of V. For every $r \in D$ define $A(r) := \sum_{i\in\mathbb{N}} s_n(r) \cdot V_n$ and $A(1) := V$. The $A(r)$ are well-defined as they are finite sums. Now we define the functional $p : V \to \mathbb{R}$, $p(v) := inf\{r | v \in A(r)\}$ and $d(v,w) := p(v-w)$. This is the metric we wanted to obtain. To prove the properties of d, we irst show

Lemma 3.2 *We have $A(a) + A(b) \subset A(a+b)$ for all $a, b, a+b \in D$.*

Proof of the lemma: We use the following fact for dyadic numbers: If $a, b, a+b \in D$ and $M := inf\{s_n(a+b) \neq s_n(a) + s_n(b)\}$, then $s_M(a) = 0 = s_M(b)$ and $s_M(a+b) = 1$. Now, if $M = \infty$, i.e., if $s_n(a+b) = s_n(a) + s_n(b)$ for all $n \in \mathbb{N}$, then of course $A(a) + A(b) = A(a+b)$ by definition. Otherwise, we get

$$A(a) \subset \sum_{i=1}^{M-1} s_i(a)V_i + s_{M+1}(a)V_{M+1} + \sum_{i=M+2}^{\infty} s_i(a)V_i$$

Now it is easy to see that for $i(1)...i(K) > J$, $\sum_{j=1}^{k} V_{i(j)} \subset V_J$ **(exercise (4)!)**, thus

$$A(a) \subset \sum_{i=1}^{M-1} s_i(a)V_i + s_{M+1}(a)V_{M+1} + s_{M+1}(a)V_{M+1},$$

and analogously

$$A(b) \subset \sum_{i=1}^{M-1} s_i(b)V_i + s_{M+1}(b)V_{M+1} + s_{M+1}(b)V_{M+1}.$$

Now as $s_i(a+b) = s_i(a) + s_i(b)$ for all $i < M$, we get

$$A(a) + A(b) \subset \sum_{i=1}^{M} s_i(a+b)V_i + V_M \subset A(a+b).$$

($\square$)

Now, if $x < y, x, y \in D$ then we have

$$A(x) \subset A(x) + A(y-x) \subset A(y), \tag{2}$$

the first inclusion being true as every $A(t)$ contains $0 \in V$. Now let any $\epsilon > 0$ be given, then the density of the dyadic numbers in $[0,1]$ implies that there are $r, s \in D$ with

$$p(a) < r, p(b) < s, r + s < p(a) + p(b) + \epsilon.$$

Therefore $p(a+b) < r+s < p(a)+p(b)+\epsilon$ for all $\epsilon > 0$ and therefore $p(a+b) < p(a)+p(b)$, which is equivalent to the triangle equality for d. As the V_n are starshaped, we have $p(-v) = p(v)$, which implies symmety of d. If $v \neq 0$, then because of Hausdorffness there is a V_n with $v \notin V_n$, and thus $d(v,0) \leq 2^{-n}$, which implies faithfulness of d, so d is a proper metric whose translation-invariance is trivially seen. It generates the original topology in V: The d-balls satisfy $B_r(0) = \bigcup_{s<r} A(s)$, so given an $n \in \mathbb{N}$, then Equation 2 implies that for every $r < 2^{-n}$ we have $B_\delta(0) \subset V_n$. As the $A(r)$ are starshaped, so are the balls. For the locally convex case show that the balls can be chosen to be convex: **Exercise (5)**. **Hint:** Use the description of the balls above and the ordering of the A_r! $\square$

Now we will give an example of a non-metrizable tvs and a motivation of its usefulness. Consider a nonempty open set $U \subset \mathbb{R}^n$. Set $C_c^\infty(U,K) := \{f \in C^\infty(U,K) | \mathrm{supp}(f) \text{ compact}\}$ with the usual C^∞-topology τ defined by the series of C^i norms supremized over U. Equally, we could choose a compast exhaustion K_n of U and consider the series of C^i seminorms supremized on K_i and the associated topology σ.

Lemma 3.3 *Given a compact set $A \subset U$, then on $C_c^\infty(A,K)$, the subspace topologies to τ and σ coincide.*

Proof: Exercise (6) $\square$

As we have already seen, this topology is locally convex, metrizable, but not complete. As often completeness is more important than metrizability, we will now construct a sequentially complete and locally convex vector topology on $C_c^\infty(U,K) =: D$ which will then turn out not to be metrizable.

For every compact $A \subset U$, denote the topology of $C^\infty(A,K)$ by t_A. Define t_n as the set of all convex and starshaped subsets N of D which satisfy $N \subset C^\infty(A,K) \in t_A$ for all compact A. These will be a base of our neighborhoods of zero (plus the empty set), and we define the topology t on D as arbitrary unions of $\{t_n + d | d \in D\}$. To prove that this defines indeed a topology, we have to show stability under finite intersections: for two open sets U_1, U_2 we will show that for every $x \in U_1 \cap U_2$ there is a zero-neighborhood N with $x + N \subset U_1 \cap U_2$. The definition of t implies that there are $x_i \in D$ and zero-neighborhoods N_i with $x \in x_i + N_i$. Now choose a compact set A s.t. $C^\infty(A,K) \ni x_1, x_2, x$. As the $C^\infty(A,K) \cap N_i$ are open in $C^\infty(A,K)$, we can find an $r > 0$ with $x - x_i \in (1-r)N_i$, and as the W_i are convex, we get

$$x - x_i + rN_i \subset (1-r)N_i + N_i = N_i.$$

Therefore $x + rN_i \subset x_i + N_i \subset U_i$, and with the definition $N := r \cdot (N_1 \cap N_2)$ we get $x + N \subset U_1 \cap U_2$ as required.

Theorem 3.4 (D,t) *is a Hausdorff tvs.*

Proof. For Hausdorffness it is sufficient to find a point v such that $\{v\}$ is closed. So let $v, w \in D$ be given and define $W_{vw} := \{f \in D : ||f||_{C^0} < ||v - w||_{C^0}\}$. This is a convex starshaped subset whose intersection with all $C^\infty(A,K)$ for compact subsets A is open, thus it is a neighborhood of zero, and $w + W_{vw}$ does not contain v. Therefore (D,t) is Hausdorff. Continuity of the vector addition holds as, for element U of the basis of t, the set $\frac{1}{2}U$ is a zero-neighborhood as well, and we get $(v_1 + \frac{1}{2}U) + (v_1 + (v_2 + \frac{1}{2}U) \subset v_1 + v_2 + U$ because of convexity of U. Scalar multiplication is continuous as, for $r, s \in K$,

$$rv_1 - sv_2 = r(v_1 - v_2) + (r - s)v_2.$$

We are looking for a condition on $r - s$ and $v_1 - v_2$ for which the above is contained in an element W of the above local basis of t. There is an $\epsilon > 0$ with $\epsilon v_2 \in \frac{1}{2}W$. Then with $c := \frac{1}{2}(|s| + \epsilon)^{-1}$, convexity and starshapedness of W imply that for $|r - s| < \epsilon$ and $v_1 - v_2 \in cW$ we have $rv_1 - sv_2 \in W$. $\square$

Theorem 3.5 *For every compact set $K \subset U$, the subspace topology of $D_K \subset (D,t)$ coincides with the topology τ_K. The tvs (D,t) is sequentially complete.*

Proof. Let $V \in t$ and $f \in D_K \cap V$ be given. By definition of t, there is a W from the basis t_n such that $f + W \subset V$, thus $f + (D_k \cap W) \subset D_k \cap V$. As $D_k \cap W$ is open in D_k, it follows that $D_k \cap V \in \tau_K$, so τ_K is finer than the subspace topology. Conversely, suppose $E \in \tau_K$. We have to show that there is a $V \in t$ with $E \supset D_K \cap V$. By definition of the topology τ_K there is a $|| \cdot ||_N$-ball in E for some $N \in \mathbb{N}$ which is the intersection of the corresponding $|| \cdot ||_N$-ball in D with D_k.

For the second assertion, we want to prove first that every s-bounded subset E of D is contained in some D_K. So consider a subset $E \subset D$ not contained in any D_K. Then there are $f_n \in E$ and points $x_n \in U$ without a limit point in U with $f_n(x_n) \neq 0 \quad \forall n \in \mathbb{N}$. Define W as the set of all $f \in D$ with $|f(x_m)| < m^{-1}|f_m(x_m)|$ for all $m \in \mathbb{N}$. As every K contains only finitely many of the x_n, $D_K \cap W \in \tau_K$, thus W is an element of the base t_n. But as $f_m \notin mW$, E is not bounded. Now, as every Cauchy sequence is s-bounded, it lies in some D_K. As the subspace topology opf the latter coincides with the complete τ_K topology, it has a limit. □

Theorem 3.6 *D is not metrizable.*

Proof. Choose a countable compact exhaustion $K_n \subset K_{n+1} \to U$. As every $D_n := D_{K_n} \subset D$ is complete, it is closed according to Theorem 23 of the first part, and it is easy to see that $\mathrm{int}_D(\overline{D_n}) = \mathrm{int}_D(D_n)$ is empty, therefore D is meager in itself. D is sequentially complete, so if it were metrizable, it would be complete metrizable and therefore nonmeager in itself according to Baire's Theorem. □

Definition 3.7 *For a real number K, a metric vector space is called* **scalar-bounded by** *K iff $d(\rho \cdot v, 0) \leq K\rho d(v, 0)$ for every $\rho \geq 1$.*

Remark. The triangle inequality implies that every Fréchet space with star-shaped balls is scalar-bounded by 2. However, even in finite-dimensional metric vector spaces, balls do not have to be star-shaped. As an example, consider the real line with the metric $d(r, s) := \Phi(|r - s|)$ with $\Phi(x) := x$ for $0 \leq x \leq 1$, $\Phi(x) := 1 - (x - 1)/2$ for $1 \leq x \leq 2$ and $\Phi(x) := 1/2 + (x - 2)/3$ for $x \geq 2$. as $\Phi(x \pm y) \leq \Phi(x) + \Phi(y)$, the metric d satisfies the triangle inequality, but the balls with radius $1/2 \leq r \leq 1$ are not starshaped and not even connected in this example.

Theorem 3.8 *In a metric vector space with starspaped balls, every s-bounded subset is bounded. In a normed vector space, s-bounded subsets are precisely the bounded subsets.*

Proof. In a metric vector space, for every s-bounded subset A there is a $K > 0$ with $A \subset KB_1(0) \subset B_{2K}(0)$ from the above. The statement for normed vector spaces is even easier. □

Theorem 3.9 *(1) A closed subspace of a Fréchet space resp. (complete) metric vector space is again a Fréchet space resp. (complete) metric vector space, scalar-bounded by the same constant.*

(2) A quotient of of a Fréchet space resp. (complete) metric vector space by a closed subspace is again a Fréchet space, scalar-bounded by the same constant.

(3) The direct sum of finitely many Fréchet spaces resp. (complete) metric vector spaces is again a Fréchet space resp. a (complete) metric vector space, scalar-bounded by the maximum of the bounds. Equally, countable products of Fréchet spaces are Fréchet spaces.

Proof. (i) Restrict the metric to the subspace and consider the relative topology of the closed subspace. Convex sets stay convex as intersected with a linear subspace. The scalar bound is trivial.

(ii) Let us call the closed subspace U and the surrounding Fréchet space X. Define the new metric d' by $d'(v, w) := min_{c \in U} d(v + c, w) = min_{c,d \in U} d(v + c, w + d)$ (the last equation is valid because of the invariance of d under translations). This metric generates the quotient topology. Now for every Cauchy sequence in X/U we have to find a Cauchy sequence of representatives in X. Thus choose a $M_\epsilon \in \mathbb{N}$ s.t. for all $m, n > M$ we have $d'([v_m],[v_n]) = min_{c \in U} d(v_m, v_n + c) < \frac{\epsilon}{3}$. Then choose a $m(0) > M$, a representative $v_{m(o)}$ and a sequence of vectors $c_n \in U$ with $d(v_{m(0)}, v_n + c_n) < \frac{\epsilon}{2}$. Then using the triangle inequality we see that for $\tilde{v}_n := v_n + c_n$ we have $d(\tilde{v}_k, \tilde{v}_l) < \epsilon$. Now modify the sequence of representatives successively this way for $\epsilon = \frac{1}{n}$ for all $n \in \mathbb{N}$. this converges and leaves us with a Cauchy sequence in X. For the scalar bound and for $\rho \geq 1$ take $\tilde{c} := \rho \cdot c$ in the definition of the distance.

(iii) In the finite case, let d_1, d_2 two metrics, we choose a continuous concave function $\Delta : \mathbb{R}^2 \to \mathbb{R}$ (e.g. $x_1 + x_2$ or $\sqrt{x_1^2 + x_2^2}$) and define the new metric $d' := \Delta \circ (d_1, d_2)$. For the scalar bound use concavity of Δ. In the countable case we have to define a metric on $F_1 \times F_2 \times ...$, this can be done by $d(s^1, s^2) := \sum_{i=1}^\infty \psi_i(d_i(s_i^1, s_i^2))$ for a supernice sequence of functions ψ.

Now the only remaining point is completeness; in the subspace case we already saw it earlier, the product case is (almost) trivial and forms together with the second case an easy **exercise(7)** □

Theorem 3.10 (metric Hahn-Banach Theorem) *Let F be a Fréchet space, $G \subset F$ a subspace and $\lambda : G \to \mathbb{R}$ a continuous linear map. Then there is a continuation of λ to a continuous linear map $F \to \mathbb{R}$. If we fix a Fréchet metric d with respect to which λ is bounded on G by R, we can choose a continuation bounded by R as well. In particular, for every vector $f \in F$, there is a continuous linear functional λ on F with $\lambda(f) \neq 0$.*

Proof. The proof is in complete analogy to the Banach case: Apply the algebraic Theorem of Hahn-Banach $p(x) = sup_{u \in U} \frac{||l(u)||}{d(u,0)} \cdot d(x, 0)$. □

In all the examples seen until now the metric could be constructed by a countable family of seminorms. This is a general feature of Fréchet spaces:

Theorem 3.11 *Let F be a Fréchetable space. Then there is a $\mathbb{N}$-family of continuous seminorms $||\cdot||_i$ on F whose balls $B^i_\epsilon(x) := \{y \in X : ||y - x||_i < \epsilon\}$ are a basis of the topology of F. Therefore the topology of F can be generated by the metric*

$$D_\alpha(f, g) := \sum_{i=1}^\infty \alpha_n \Phi(||f - g||_i) \tag{3}$$

where α is an arbitrary positive sequence converging to 0, and as well by

$$d_\alpha(f, g) := \sup_{i \in \mathbb{N}} \alpha_n \Phi(||f - g||_i). \tag{4}$$

It can be assumed w.r.o.g. that the series of seminorms is monotonous in the sense that for every fixed vector $v \in F$ we have $||v||_i \leq ||v||_{i+1}$ for all natural i.

Proof. Choose a Fréchet metric d, consider $B^d_{\frac{1}{i}}(0)$ and define the seminorms as the so-called Minkowski functionals

$$||v||_i := inf\{\lambda > 0 | \frac{1}{\lambda} \cdot v \in U_i\}.$$

where we choose convex subsets $U_i \subset B^d_{\frac{1}{i}}(0)$. These Minkowski functionals are subadditive, as for $\frac{1}{\lambda} f, \frac{1}{\mu} g \in U_i$ we have also $\frac{1}{\lambda + \mu}(f + g) \in U_i$ as a convex combination.

Continuity is an easy consequence of subadditivity. Finally, Cauchy sequences w.r.t. all $||\cdot||_i$ are Cauchy sequences for the metric. $\square$

It should be stated, however, that if the metric d was already given as a sum of seminorms as in Equation (3), the Minkowski functionals will *not* give us back the original seminorms.
Let us come back to our examples. Comparing Example 2 and Example 3 we notice that in Example 2 none of the seminorms we used is a norm while in Example 3 any of the seminorms is a norm. The question could arise whether there is *any* continuous norm on $F^{\mathbb{N}}$. This question is answered negatively in the following theorem.

Theorem 3.12 *The Fréchet space $F^{\mathbb{N}}$ does not have a continuous norm.*

Proof. Let us assume the existence of a continuous norm ν. Then we consider a ball $B_R^\nu(0)$. On one hand, this ball cannot contain any nontrivial subspace of $F^{\mathbb{N}}$, as ν is homogeneous w.r.t. the multiplication by positive numbers. But on the other hand, the ball is open because of continuity of ν, so it contains a finite intersection of elements $B_R^{||\cdot||_i}(0)$ of the basis of the topology. But an intersection of the balls for the seminorms $||\cdot||_{i_1},...||\cdot||_{i_n}$ contains the subspace $\{x \in F^{\mathbb{N}} | x_1 = ... = x_m = 0\}$ where $m = max\{i_1, ...i_n\}$, a contradiction. $\square$

Definition 3.13 *A topological space T is called* **paracompact** *if and only if every open covering of T contains a locally finte subcovering, i.e. every point of T has a neighborhood intersecting only finitely many of the open sets of the covering.*

Theorem 3.14 (by A.H. Stone, through Abraham/Marsden/Ratiu) *Every metric space is paracompact.*

Proof. Let U_i, $i \in I$, be a n open covering of a metric space (X, d). Then define $U_{n,\alpha} := \{x \in U_\alpha | d(x, X \setminus U_n) \geq 2^{-n}\}$, then the triangle inequality implies that $d(U_{n,\alpha}, X \setminus U_{n+1,\alpha}) \geq 2^{-(n+1)}$. Then set

$$V_{n,\alpha} := \bigcup_{\beta \in I : U_\beta \subset U_\alpha} U_{n+1,\beta}.$$

Now for $U_\gamma \subset U_\delta$ we have $V_{n,\gamma} \subset X \setminus U_{n+1,\delta}$, thus if $U_\gamma \subset U_\delta$ or $U_\delta \subset U_\gamma$, we have $d(V_{n,\gamma}, V_{n,\delta}) \geq 2^{-(n+1)}$. Finally, define

$$W_{n,\alpha} := \{x \in X | d(x, V_{n,\alpha}) < 2^{-(n+3)}\}.$$

Then $d(W_{n,\alpha}, W_{n,\beta}) \geq 2^{-(n+2)}$. Therefore for $n \in \mathbb{N}$ fixed, every point $x \in X$ lies in at most one element of $\{W_{n,\alpha} | \alpha \in A\}$, then for higher $m > n$ there cannot be a $\beta \neq \alpha$ with $x \in W_{\beta,m}$. In the same time, $d(x, X \setminus U_\alpha)$ gives a bound of the n for which $x \in W_{\alpha,n}$. Thus the family $\{W_{\alpha,n} | \alpha \in I, n \in \mathbb{N}\}$ is a locally finite refinement. $\square$

Often instead of with a single continuous function one deals with collections of those (e.g., sequences). It is convenient to extend the notion of continuity to those sets of functions:

Definition 3.15 *Let X, Y be tvs. A subset A of $CL(X, Y)$ is called* **equicontinuous** *if for every neighborhood U of $0 \in Y$ there is a neighborhood of $V \in X$ with $A(V) := \bigcup_{f \in A} f(V) \subset U$.*

Theorem 3.16 (Banach-Steinhaus Theorem) *Let X and Y be tvs, A a subset of $CL(X,Y)$. Let B be the subset of X whose points b have s-bounded orbits $Ax := \{fx|f \in A\}$. If B is nonmeager in X, then $B = X$ and A is equicontinuous.*

Proof. We choose balanced zero-neighborhoods W, U in Y with $\overline{U} + \overline{U} \subset W$ and define $E := \bigcap_{f \in A} f^{-1}(\overline{U})$. For every $x \in B$ there is a $K \in \mathbb{R}$ with $A(x) \in K\overline{U}$, therefore $x \in KE$, so $\bigcup_{K \in \mathbb{N}} KE \supset B$. As B is nonmeager, some NE has to be nonmeager, and, by the bi-continuity of the scalar multiplication with N, E itself is nonmeager. But by continuity of each $f \in A$, it is an intersection of closed sets and therefore closed, thus $\emptyset \neq int(E) \ni p$, so $W := p - E$ is a neighborhood of 0 in X, and

$$A(W) = Ap - A(E) \subset \overline{U} - \overline{U} \subset W.$$

Therefore A is equicontinuous. Now given a point $x \in X$ and a zero neighborhood N in Y, take a zero neighborhood M in X with $A(M) \subset N$, then by continuity of scalar multiplication we find an $r > 0$ with $rx \in M$, therefore $Ax \subset r^{-1}N$. Therefore x has a bounded orbit, and $B = X$. □

As by Baire's Theorem, complete metrizable tvs are nonmeager in themselves, we get as corollary

Theorem 3.17 *Let X be a complete metrizable tvs and Y a tvs, let $A \subset CL(X,Y)$ such that all A-orbits are bounded. Then A is equicontinuous.* □

Exercise (8): Let F be a Fréchet space and $U \subset F$ convex and open. Show that every continuous $f : U \to K$, where $K \subset U$ is a compact set, has a fix point.

Exercise(9): Show that there is an $f \in C^0([0,1], \mathbb{R})$ such that for all $x \in [0,1] =: I$

$$f(x) = \int_0^1 sin(x + f^2(t))dt.$$

Hint: Denote the RHS by $(Af)(x)$, show that $S := \{Af|f \in C(I)\} \subset C(I)$ is uniformally bounded and equicontinuous and that therefore $\overline{S}$ is compact. Then apply the previous exercise.
A map f from a topological space S to a topological space T is caled **open** if $f(U)$ is open in T for every open set U in S.

Theorem 3.18 (Open mapping theorem) *Let X be a complete metrizable tvs, Y a tvs, $L \in CL(X,Y)$ and $L(X)$ nonmeager in Y. Then L is open and surjective and Y is complete and metrizable.*

Proof. Surjectivity follows from openness as the only open subset of a tvs is the tvs itself. Now let V be a zero neighborhood of in X. We will show that $L(X)$ is a zero neighborhood in Y. To this purpose, define a translation-invariant compatible metric on X, choose $r > 0$ with $B_r(0) \subset V$ and consider, for all $n \in \mathbb{N}$, the neighborhoods $B(n) := B_{2^{-n}r}(0)$. We will show that, for all $n \in \mathbb{N}$

$$0 \in \mathrm{int}(\overline{f(B(n+1))}) \subset \overline{f(B(n))} \subset f(V). \tag{5}$$

The fact that $B(n+1) - B(n+1) \subset B(n)$ and continuity of the vector addition imply

$$\overline{f(B(n+1))} - \overline{f(B(n+1))} \subset \overline{f(B(n+1)) - f(B(n+1))} \subset \overline{f(B(n))},$$

so for the first inclusion of sets in Equation 5 we have to show that $\mathrm{int}(\overline{f(B(n+1))})$ is nonempty. Then we proceed as in the proof of the Banach-Steinhaus Theorem: As $B(n+1)$ is a zero neighborhood, $f(X) = \bigcup_{i\in\mathbb{N}} if(B(n+1))$, and therefore there is a nonmeager $if(B(n+1))$. Now, as scalar multiplication is a homeomorphism, $f(B(n+1))$ is nonmeager itself, thus its closure has nonempty interior.
For the second inclusion we choose inductively points $y_i \in \overline{f(B(i))}$ using that each $\overline{f(B(n))}$ is a zero neighborhood, such that we can choose a point p_{n+1} from the nonempty $(y_n - \overline{f(B(n+1))}) \cap f(B(n))$. Then we define $y_{n+1} = y_n - p_{n+1} \in \overline{f(B(n+1)))} \cap (y_n - f(B(n)))$. If we choose some preimages $x_n \in B(n)$ of the p_n then $d(x_n, 0) < 2^{-n}r$ for all $n \in \mathbb{N}$, thus $\sum_{i=1}^{\infty} x_n =: x$ exists as the partial sums form a Cauchy sequence, and $d(x, 0) < r$, so $x \in V$. Now

$$f(x) = f(\lim_{n\to\infty} \sum_{i=1}^{n} x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} (y_i - y_{i+1}) = y_1 - \lim n \to \infty y_n = y_1$$

by continuity and linearity of f. This shows the second inclusion for $n = 1$ (and the other cases by $B(n+1) \subset B(n)$) and thus the openness of f.
Now by Theorem 3.9 by closedness of the kernel K of f it is obvious that X/N with the quotient topology is a complete metrizable tvs. Finally, the openness of f implies easily a homeomorphism between X/N and Y. □

Again with Baire's Theorem we get as a corollary:

Theorem 3.19 *Let X and Y be complete metrizable tvs and $L \in CL(X,Y)$ be surjective, then L is open.*

Theorem 3.20 *Let F be a vector space and t_1, t_2 two Fréchetable topologies on them. If t_1 is finer than t_2, then they are equal.* □

Example: Hamilton ([**?**]) gives an example of a closed subspace of a Fréchet space which is not topologically complemented. So take $F := C^\infty([0,1])$ which contains the space G of 1-periodic real functions on the real line, $C_1^\infty(\mathbb{R})$ by the restriction ρ on the unit interval. If we define

$$p : C^\infty([0,1]) \to \mathbb{R}^{\mathbb{N}}, f \mapsto \{D^j f(2\pi) - D^j f(0)\}_{j\in\mathbb{N}}$$

we get the short exact sequence

$$\{0\} \to C_1^\infty(\mathbb{R}) \to^{\rho} C^\infty([0,1]) \to^{p} \mathbb{R}^{\mathbb{N}} \to \{0\}$$

Thus the quotient of $C^\infty([0,1])$ by $C_1^\infty(\mathbb{R})$ is homeomorphic to $\mathbb{R}^{\mathbb{N}}$. As the latter one does not have any continuous norm, there cannot be a continuous linear isomorphism between $\mathbb{R}^{\mathbb{N}}$ and any closed subspace of F. Therefore the above sequence does not split, and G is not topologically complemented in F.

This behaviour is not exceptional which is shown by the following theorem we quote from Köthe's book:

Theorem 3.21 (cf. [?], p. 435) *Let F be a Fréchet space with a continuous norm which is not Banach. Then there is a closed subspace $H \subset F$ with $F/H \cong \mathbb{R}^{\mathbb{N}}$, thus H is not topologically complemented in F.*

But at least simple subspaces of tvs are topologically complemented:

Theorem 3.22 *Let V be a Hausdorff tvs. Then*

(1) Every finite-dimensional subspace of V is closed.

(2) Every closed subspace $G \subset V$ with $codim(G) = dim(V/G) < \infty$ is topologically complemented in V (by each of its algebraic complements).

(3) If V is locally convex, every finite-dimensional subspace of F is topologically complemented.

(4) If V is complete and metrizable, every linear isomorphism between the direct sum of two closed subspaces and F, $G \oplus H \to F$, is a homeomorphism.

Proof. The first part is only a rewording of a result we have seen already. For the second part, take any algebraic complement C of G, it is finite-dimensional and therefore closed. The projection P of V onto C with kernel G is the composition of the quotient map $q: V \to V/G$, the linear bijection $B: V/G \to C$ between vector spaces linearly homeomorphic to a K^n and the imbedding $C \to V$, therefore it is continuous. Therefore C is a topological complement. The third part can be proven by choosing a basis a_i for the subspace S, then by Hahn-Banach extend the associated linear functionals to $A_i \in CL(V, \mathbb{R})$ and then to define $C := \bigcap_{i=1}^n \ker(A_i)$ as a complementary subspace. (4) is an **exercise (10)**. □

Theorem 3.23 (Meise/Vogt) *Let F be a Fréchetable space, let $(e_i)_{i\in\mathbb{N}}$ be a countable topological basis for F, let $(||\cdot||_i)_{i\in\mathbb{N}}$ be a monotonous series of seminorms generating the topology of F (e.g. the sequence of Minkowski functionals). Then for every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ and a $C > 0$ such that for every $v \in F$ we have*

$$\sup_{k\in\mathbb{N}} ||\sum_{i=1}^{k} v_i \cdot e_i||_m \leq C||v||_n,$$

where $v_i := \xi_i(v)$ are the unique coefficients of v w.r.t. the basis e_i.

Proof. We define, for all natural n,

$$||v||'_n := \sup_{k\in\mathbb{N}} ||\sum_{i=1}^{k} v_i e_i||_n.$$

Obviously, $||\cdot||'_n$ defines a monotonous sequence of seminorms on F with $||\cdot||'_n \geq ||\cdot||_n$, thus the locally convex metrizable topology t_2 defined by the sequence of the $||\cdot||'_n$ is finer than the original topology t_1 on F. Now we show that t is a complete topology on F, the rest follows by the corollary of the open mapping theorem. Our basic estimate is the following: for every $x \in F$ and every $n, k \in \mathbb{N}$ we have

$$\begin{aligned}
|x_k| \cdot ||e_k||_n = ||x_k \cdot e_k||_n &= ||\sum_{i=1}^{k} x_i \cdot e_i - \sum_{i=1}^{k-1} x_i \cdot e_i||_n \\
&\leq ||\sum_{i=1}^{k} x_i \cdot e_i|| + ||\sum_{i=1}^{k-1} x_i \cdot e_i|| \\
&\leq 2\sup_{k\in\mathbb{N}}(||\sum_{i=1}^{k} x_i \cdot e_i||_n) = 2||x||'_n.
\end{aligned}$$

Now let v^j be a t_2-Cauchy sequence in F, then first we want to prove that, for all $k \in \mathbb{N}$, the sequence $j \mapsto \xi_k(v_j)$ is a Cauchy sequence in $\mathbb{K}$. For this, choose an $n \in \mathbb{N}$ with $||e_k||_n > 0$, then the above basic estimate $|\xi_k(v_\nu) - \xi_k(v_\mu)| \cdot ||e_k||_n \leq 2||v_\nu - v_\mu||'_n$ implies that $j \mapsto \xi_k(v_j)$ is a Cauchy sequence, and, by completeness of $\mathbb{K}$, has a unique limit x_k. It remains to show that $x := \lim_{l\to\infty} \sum_{i=1}^{l} x_j e_j$ exists and that

$\lim_{j\to\infty} v_j = v$ in the topology t_2. To this purpose, let $n \in \mathbb{N}$ be fixed until further notice. As for all $i \in \mathbb{N}$, the sequence $j \mapsto \xi_i(v_j)$ converges, for all $k \in \mathbb{N}$ there is a $\nu \in \mathbb{N}$ such that for all $\mu > \nu$ we have

$$||\sum_{i=1}^{k} \xi_i(v_\mu)e_i - \sum_{i=1}^{k} x_i e_i||_n \leq \epsilon. \tag{6}$$

Thus for all $k, p \in \mathbb{N}$ we have

$$\begin{aligned}
||\sum_{i=k+1}^{k+p} x_i e_i||_n &= ||-\sum_{i=1}^{k+p} \xi_i(v_\nu)e_i + \sum_{i=1}^{k+p} x_i e_i - \sum_{i=1}^{k} x_i e_i + \sum_{i=1}^{k+p} \xi_i(v_\nu)e_i||_n \\
&\leq ||\sum_{i=1}^{k+p} x_i e_i - \sum_{i=1}^{k+p} \xi_i(v_\nu)e_i||_n + ||\sum_{i=1}^{k} x_i e_i - \sum_{i=1}^{k} \xi_i(v_\nu)e_i||_n + ||\sum_{i=k+1}^{k+p} \xi_i(v_\nu)e_i||_n \\
&\leq 2\epsilon + 2||\sum_{i=k+1}^{k+p} \xi_i(v_\nu)e_i||_n.
\end{aligned}$$

As the series $j \mapsto \sum_{i=1}^{j} \xi_j(v_\nu)e_i$ converges to v_ν and as all of this holds for an arbitrary $n \in \mathbb{N}$, we have that $j \to \sum_{i=1}^{j} x_i e_i$ is a t_1-Cauchy sequence and converges therefore to, say, x, whose coefficients in turn are given uniquely by x_i as the e_i form a basis. Then the t_2-convergence of $j \to v_j$ to x is implied by Equation 6. □

As a corollary, we obtain

Theorem 3.24 *Every countable topological basis of a Fréchet space is continuous (and therefore Schauder).*

We have seen in the preceeding chapter that for a continuous function f between Hausdorff tvs, a necessary condition for f to be continuous is the graph G_f to be closed. Now in closed metrizable tvs this condition is also sufficient for linear maps:

Theorem 3.25 (Closed Graph Theorem) *Let F and H are complete metrizable tvs. A linear map $L : F \to H$ is continuous if and only if its graph $G := G_L$ is closed.*

Proof. First we infer from Theorem 3.9 that for d_F resp. d_H being a compatible metric in F resp. H, the metric sum $d_F + d_H$ is a compatible metric for the product topology on $F \times H$ and the component-wise addition and scalar multiplication are continuous. As L is linear, its graph G is a linear subspace of $F \times H$. If it is closed, Theorem 23 of the first part tells us that it is complete, and Theorem 3.9 tells us that it is metrizable. Now let $pr_i, i = 1, 2$ be the projections of $F \times H$ onto its components. Then $p := pr_1|_G$ is a continuous linear bijective map from the complete metrizable tvs G to the complete metrizable tvs F. The open mapping theorem implies that p is open, thus $p^{-1} : F \to G$ is continuous, thus $L = pr_2 \circ p^{-1}$ is continuous. □

Definition 3.26 *Let F be a Fréchet space, let p be a seminorm on F. Then the* **local Banach space to** *p is defined as $F_p := (E/N(p), p)\hat{}$ for $N(p)$ the null space of p and $\hat{}$ the completion.*

Finally we present one of the most important fix point theorems in partial differential equations:

Theorem 3.27 (Banach's fix point Theorem) *Let* (X, d) *a complete metric space and* $f : X \to X$ *a contraction with contraction factor* $\rho < 1$*. Then* f *has a unique fix point* x_f *in* X*. It is the limit of the recursive sequence* $x_0 \in X$ *arbitrary,* $x_{n+1} = f(x_n)$*. The distance to the solution decreases like*

$$d(x_n, x_f) \leq \frac{\rho^n}{1-\rho} d(x_0, x_1).$$

Exercise (11): Please *everyone* of You look for a **proof** in the literature or, better even, prove it Yourself, as this theorem is a cornerstone of analysis.